Online Appendix to
"Does Easing Monetary Policy Increase Financial Instability?"
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This Appendix describes the numerical solution of the model in Cesa-Bianchi and Rebucci (2016). We closely follow the approach proposed by Jeanne and Korinek (2010). First, describe the first order conditions of the model. Second, we derive the combination of parameter values that make the collateral constraint binding with a positive probability. Finally, we show how to solve for the optimal allocations of the decentralized equilibrium and the social planner equilibrium.

First order conditions. We solve for the equilibrium backward, as in Jeanne and Korinek (2010). As stated in the main text, in period 1 consumers maximize their utility:

$$u(c_{i,0}) + u(c_{i,1}) + c_{i,2},$$

where, for simplicity, we assume a unitary discount factor. The period utility function, $u(\cdot)$, is a standard CES function:

$$u(c) = \frac{c^{1-\varrho}}{1-\varrho}.$$

Consumers are subject to the following budget constraint:

$$\begin{cases} c_{i,0} = b_{i,1} + (1-\theta_{i,1})p_0, \\ c_{i,1} + b_{i,1}R_{L1} = e + b_{i,2} + (\theta_{i,1} - \theta_{i,2})p_1 + \pi_{i,1}, \\ c_{i,2} + b_{i,2}R_{L2} = \theta_{i,2}y + \pi_{i,2}, \end{cases}$$

and the following collateral constraint:

$$b_{i,2} \leq \theta_{i,1}p_1.$$

The problem for the representative consumer therefore is:

$$\mathcal{V}_1 = \max_{b_2,\theta_2} \left\{ u\Big(e + b_2 + (\theta_1 - \theta_2)p_1 + \pi_1 - b_1 R_{L1} \Big) + \theta_2 y + \pi_2 - b_2 R_{L2} - \lambda(b_2 - \theta_1 p_1) \right\},$$

where net worth $(e - b_1 R_{L1})$ is taken as given. The first order conditions are:

$$\begin{cases} FOC(b_2): & u'(c_1) = R_{L2} + \lambda, \\ FOC(\theta_2): & p_1 = y/u'(c_1). \end{cases}$$

In period 0, consumers solve the following problem:

$$\max_{b_1} \left\{ u(b_1) + \mathbb{E}_0 \left[\mathcal{V}_1 \right] \right\},$$

where we make use of the fact that, in equilibrium, $\theta_t = 1$. The maximization yields:

$$u'(c_0) = R_{L1} \mathbb{E}_0 \big[u'(c_1) \big].$$

The first order conditions of the competitive equilibrium (CE) therefore are:

$$\begin{cases} FOC(b_1): & u'(c_0) = R_{L1} \mathbb{E}_0 \big[u'(c_1) \big], \\ FOC(b_2): & u'(c_1) = R_{L2} + \lambda, \\ FOC(\theta_2): & p_1 = y/u'(c_1). \end{cases}$$

When the economy is not constrained ($\lambda = 0$) the model has the following close form solution:

$$\begin{cases} u'(c_1) = R_{L2} \\ u'(c_0) = \mathbb{E}_0 \left[R_{L2} R_{L1} \right], \\ p_1 = \frac{y}{R_{L2}} \end{cases} \implies \begin{cases} c_1^* = (R_{L2})^{-\frac{1}{\varrho}} \\ c_0^* = b_1^* = (R_{L2} R_{L1})^{-\frac{1}{\varrho}}, \\ p_1^* = \frac{y}{R_{L2}}. \end{cases}$$

Moreover, by definition, the collateral constraint must hold when the economy is not constrained:[1]

$$\underbrace{b_2^*}_{c_1^* + b_1^* R_{L1} - e} \leq \underbrace{p_1^*}_{\frac{y}{R_{L2}}},$$

which we can rewrite as:

$$e \geq e^b \quad = c_1^* + b_1^* R_{L1} - \tfrac{y}{R_{L2}}.$$

That is, whenever the endowment is above a certain threshold ($e \geq e^b$) the economy is never constrained. When the economy is constrained ($e < e^b$) consumers borrow up to the limit and maximize consumption in period 1. In this case, $b_2 = p_1$, so that:

$$c_1 + b_1 R_{L1} - e = \frac{y}{u'(c_1)}.$$

And using the fact that the utility function is in CES we have:

$$c_1 + b_1 R_{L1} - e = y c_1^{\varrho}. \tag{1}$$

Therefore, depending whether the constraint is binding or not, we can express borrowing in period 0 as:

$$b_1 = \begin{cases} (R_{L2} R_{L1})^{-\frac{1}{\varrho}} & e \geq e^b \\ \frac{y c_1^{\varrho} - c_1 + e}{R_{L1}} & e < e^b \end{cases} \tag{2}$$

We finally assume that the endowment is stochastic and follows a uniform distribution $e \sim U(\bar{e} - \varepsilon, \bar{e} + \varepsilon)$.

Assumption on parameter values. As we discussed in the text, to be able to solve the model we need to make assumptions on the value of two parameters: y and $\bar{e}$. In particular, we will consider

[1] Note here that we are assuming that profits are realized at the end of the period so that they have no effect on the borrowing constraint.

values such that the economy may be constrained for sufficiently large negative shocks, but is not constrained in the absence of uncertainty.

First, we find a condition that is necessary and sufficient for the economy to be constrained with positive probability, conditional on $e \sim U(\bar{e}-\varepsilon, \bar{e}+\varepsilon)$. We know that the economy is always unconstrained in period 1 if and only if:

$$e \geq e^b = c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}}.$$

When e is stochastic, the economy is unconstrained if and only if the above inequality holds for all possible realizations of e. So it must be the case that:

$$\begin{aligned} e - \varepsilon &\geq c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}}, \\ \bar{e} &\geq c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}} + \varepsilon. \end{aligned}$$

Therefore, when $\bar{e} < c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}} + \varepsilon$ there is positive probability that the constraint binds.

Second, we need a condition that is necessary and sufficient for the economy to be unconstrained when there is no uncertainty (i.e., $\varepsilon = 0$ and $\bar{e} = e$). When $\varepsilon = 0$, the constraint is not binding in period 1 if and only if $e = \bar{e} \geq e^b$; that is:

$$\bar{e} \geq c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}}.$$

Therefore, with no uncertainty, when $\bar{e} \geq c_1^* + b_1^* R_{L1} - \frac{y}{R_{L2}}$ the constraint never binds.

Summarizing, we choose an $\bar{e}$ such that the economy will not be constrained in the absence of uncertainty, but it may be constrained for sufficiently large negative shocks:

$$(R_{L2})^{-\frac{1}{\varrho}} + (R_{L2}R_{L1})^{-\frac{1}{\varrho}} R_{L1} - \frac{y}{R_{L2}} \leq \bar{e} < (R_{L2})^{-\frac{1}{\varrho}} + (R_{L2}R_{L1})^{-\frac{1}{\varrho}} R_{L1} - \frac{y}{R_{L2}} + \varepsilon.$$

This implies that there is a threshold for the size of the shock (ε^b) above which the collateral constraint will start to be bind with positive probability. Specifically, the collateral constraint can bind with positive probability for realizations of e in the interval $[\bar{e}-\varepsilon, \bar{e}-\varepsilon^b]$. The level of ε^b can be easily computed as:

$$\varepsilon^b = \bar{e} - e^b = \bar{e} - c_1^* - b_1^* R_{L1} + \frac{y}{R_{L2}}.$$

Competitive equilibrium. We find numerical values for consumption at time 1 (c_1) from the Euler equation $FOC(b_1)$.[2] In order to be able to solve this equation we need to find an expression for borrowing as a function of consumption in both constrained and unconstrained states, as we already did in equation (2), and then to weight those states with their probability.

Combining $FOC(b_1)$, the budget constraint, and the expression for b_1 derived earlier in equation (2) we get the following system of equations:

$$\begin{cases} b_1^{-\varrho} = R_{L1} \mathbb{E}_0 \left[c_1^{-\varrho} \right], \\ b_1 = \begin{cases} (R_{L2}R_{L1})^{-\frac{1}{\varrho}} & e \geq e^b, \\ \frac{y c_1^{\varrho} - c_1 + e}{R_{L1}} & e < e^b. \end{cases} \end{cases}$$

[2] Rember that $c_0 = b_1$ from the budget constraint.

By plugging the second equation in the first one we can write:

$$\Pr(e<e^b)\cdot\left[b_1^{-\varrho}\right]^{\text{binding}}+\Pr(e\geq e^b)\cdot\left[b_1^{-\varrho}\right]^{\text{non-binding}}=R_{L1}\mathbb{E}_0\left[c_1^{-\varrho}\right].$$

Now, by substituting for b_1, the left hand side (LHS) of this equation can be expressed as follows:[3]

$$\begin{aligned}
b_1^{-\varrho} &= \frac{1}{2\varepsilon}\int\limits_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b}\left(\frac{yc_1^{\varrho}-c_1+e}{R_{L1}}\right)^{-\varrho}de+\frac{1}{2\varepsilon}\int\limits_{\bar{e}-\varepsilon^b}^{\bar{e}+\varepsilon}R_{L2}R_{L1}de= \\
&= \frac{1}{2\varepsilon}\int\limits_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b}\left(\frac{yc_1^{\varrho}-c_1}{R_{L1}}+\frac{e}{R_{L1}}\right)^{-\varrho}de+\frac{R_{L2}R_{L1}}{2\varepsilon}\Big[e\Big]_{\bar{e}-\varepsilon^b}^{\bar{e}+\varepsilon}= \\
&= \frac{1}{2\varepsilon}\left[R_{L1}\frac{\left(\frac{yc_1^{\varrho}-c_1}{R_{L1}}+\frac{e}{R_{L1}}\right)^{-\varrho+1}}{-\varrho+1}\right]_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b}+\frac{R_{L2}R_{L1}}{2\varepsilon}\left[\varepsilon+\varepsilon^b\right] \\
&= \frac{R_{L1}^{\varrho}}{2\varepsilon\left(1-\varrho\right)}\left[\left(yc_1^{\varrho}-c_1+e\right)^{-\varrho+1}\right]_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b}+\frac{R_{L2}R_{L1}}{2\varepsilon}\left[\varepsilon+\varepsilon^b\right].
\end{aligned}$$

By equating LHS and RHS numerically, we obtain the competitive equilibrium level of consumption at time 1, where:

$$\begin{aligned}
\text{LHS} &= \frac{R_{L1}^{\varrho}}{2\varepsilon\left(1-\varrho\right)}\left[\left(yc_1^{\varrho}-c_1+\bar{e}-\varepsilon^b\right)^{-\varrho+1}-\left(yc_1^{\varrho}-c_1+\bar{e}-\varepsilon\right)^{-\varrho+1}\right]+\frac{R_{L2}R_{L1}}{2\varepsilon}\left[\varepsilon+\varepsilon^b\right] \\
\text{RHS} &= R_{L1}\mathbb{E}_0\left[c_1^{-\varrho}\right].
\end{aligned}$$

Finally, one can also derive the level of debt at time 0, by using again $FOC(b_1)$:

$$b_1=\mathbb{E}_0\left[\left(R_{L1}c_1^{-\varrho}\right)^{-\frac{1}{\varrho}}\right].$$

Social planner. The social planner problem is solved with the same strategy. The first order conditions are:

$$\begin{cases}
FOC(b_1): & u'(c_0)=R_{L1}\mathbb{E}_0[u'(c_1)+\lambda p'(c_1)], \\
FOC(b_2): & u'(c_1)=R_{L2}+\lambda(1-p'(c_1)), \\
FOC(\theta_2): & p_1=\frac{y}{u'(c_1)}.
\end{cases}$$

First we find an expression for $p'(c_1)$. From $FOC(\theta_2)$ we get:

$$p(c_1)=\frac{y}{u'(c_1)}=yc_1^{\varrho},$$

and computing the derivative:

$$p'(c_1)=\frac{\partial\left(yc_1\right)}{\partial c_1}=\varrho yc_1^{\varrho-1}.$$

Notice here that the $p'(c_1)$ is positive and decreasing. By looking at $FOC(b_1)$ for the social planner problem, we can see that the she/he borrows less than in the competitive equilibrium. In

[3]If X is uniformely distributed with $U(a,b)$, then the nth moment of X is given by $\mathbb{E}\left[X^n\right]=\frac{1}{b-a}\int\limits_a^b x^n dx$.

fact, given that λ is positive only when the constraint binds, $u'(c_1)^{SP} > u'(c_1)^{CE}$ implying that consumption and, therefore, borrowing at time 1 are lower relative to the competitive equilibrium. On the other hand, the planner increases consumption in period 1: given that $p'(c_1) > 0$, from $FOC(b_2)$ we see that $u'(c_1)^{SP} < u'(c_1)^{CE}$.

We also need a value of λ. Notice that the Lagrange multiplier of the social planner is numerically different from the one of the competitive equilibrium problem. In fact, from $FOC(b_2)$ we get

$$\lambda = \frac{c_1^{-\varrho} - R_{L2}}{1+y}.$$

Combining these two results we can compute:

$$\lambda p'(c_1) = \begin{cases} 0 & e \geq e^b, \\ \frac{\varrho y}{1+y}\left(c_1^{-1} - R_{L2} c_1^{\varrho-1}\right) & e < e^b. \end{cases}$$

We can now solve for the level of c_1. The $FOC(b_1)$ can be written:

$$b_1^{-\varrho} = R_{L1}\mathbb{E}_0\left[c_1^{-\varrho} + \lambda p'(c_1)\right].$$

The LHS is the same as before. The RHS now is:

$$\begin{aligned} &\frac{R_{L1}}{2\varepsilon}\int_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b}\left(c_1^{-\varrho} + \frac{\varrho y}{1+y}\left(c_1^{-1} - R_{L2}c_1^{\varrho-1}\right)\right)de + \frac{R_{L1}}{2\varepsilon}\int_{\bar{e}-\varepsilon^b}^{\bar{e}+\varepsilon} c_1^{-\varrho} de, \\ &\frac{R_{L1}}{2\varepsilon}\left[\left(c_1^{-\varrho} + \frac{\varrho y}{1+y}\left(c_1^{-1} - R_{L2}c_1^{\varrho-1}\right)\right)\left(\varepsilon - \varepsilon^b\right) + c_1^{-\varrho}\left(\varepsilon + \varepsilon^b\right)\right], \\ &\frac{R_{L1}}{2\varepsilon}\left[\left(\frac{\varrho y}{1+y}\left(c_1^{-1} - R_{L2}c_1^{\varrho-1}\right)\right)\left(\varepsilon - \varepsilon^b\right) + 2c_1^{-\varrho}\varepsilon\right] \end{aligned}$$

And by equalizing LHS to RHS numerically, we obtain consumption at time 1, where:

$$\begin{aligned} \text{LHS} &= \frac{R_{L1}^{\varrho}}{2\varepsilon(1-\varrho)}\left[\left(yc_1^{\varrho} - c_1 + \bar{e} - \varepsilon^b\right)^{-\varrho+1} - \left(yc_1^{\varrho} - c_1 + \bar{e} - \varepsilon\right)^{-\varrho+1}\right] + \frac{R_{L2}R_{L1}}{2\varepsilon}\left[\varepsilon + \varepsilon^b\right] \\ \text{RHS} &= \frac{R_{L1}}{2\varepsilon}\left[\left(\frac{\varrho y}{1+y}\left(c_1^{-1} - R_{L2}c_1^{\varrho-1}\right)\right)\left(\varepsilon - \varepsilon^b\right) + 2c_1^{-\varrho}\varepsilon\right]. \end{aligned}$$

Finally, we can derive the optimal expression for borrowing at time 1 from the social planner $FOC(b_1)$:

$$b_1 = \left(R_{L1}\mathbb{E}_0\left[c_1^{-\varrho} + \lambda p'(c_1)\right]\right)^{-\frac{1}{\varrho}}.$$

Crisis Probability. The crisis probability is defined as the probability that the constraint binds. Therefore:

$$\begin{aligned} &\Pr\left[b_2 > p_1\right] \\ &= \frac{1}{2\varepsilon}\int_{\bar{e}-\varepsilon}^{\bar{e}-\varepsilon^b} de = \frac{1}{2\varepsilon}\left(\varepsilon - \varepsilon^b\right). \end{aligned}$$

By using the optimality conditions and the budget constraint, this expression can be written as

$$\Pr\left[(c_1 - (e - b_1 R_{L1}) > \frac{y}{u'(c_1)}\right].$$

Now, knowing that $e = \bar{e} + \tilde{\varepsilon}$ and that $\tilde{\varepsilon} \sim \mathcal{U}(-\varepsilon, \varepsilon)$, we can write

$$\Pr\left[\tilde{\varepsilon} < \underbrace{c_1 - \bar{e} + b_1 R_{L1} - \frac{y}{u'(c_1)}}_{x}\right].$$

In particular, the probability that the constraint binds is given by:

$$\Pr\big[-\varepsilon \le \tilde{\varepsilon} < x\big] = \frac{x - (-\varepsilon)}{2\varepsilon} = \frac{c_1 - \bar{e} + b_1 R_{L1} - y/u'(c_1) + \varepsilon}{2\varepsilon}.$$

References

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