Notes on ”AGENCY COSTS, NET WORTH, AND BUSINESS CYCLE FLUCTUATIONS: A COMPUTABLE GENERAL EQUILIBRIUM ANALYSIS,” by Carlstrom and Fuerst (1997, AER)

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Chapter 1

Introduction

In the standard representative agent model financial intermediation happens costlessly and perfectly. As noted by Christiano (2005), this in part reflects the assumption that households are homogeneous:

> ”Consider, for example, the process by which physical capital is produced in the model. First, homogeneous output is produced by firms. Then, households purchase that output and use it as input into a technology that converts it one-for-one into consumption goods and new capital goods. Although one can imagine that financing is used here, that financing involves no conflict because the people applying the resources (the output goods used to produce new capital) and the people supplying the resources are the same.”

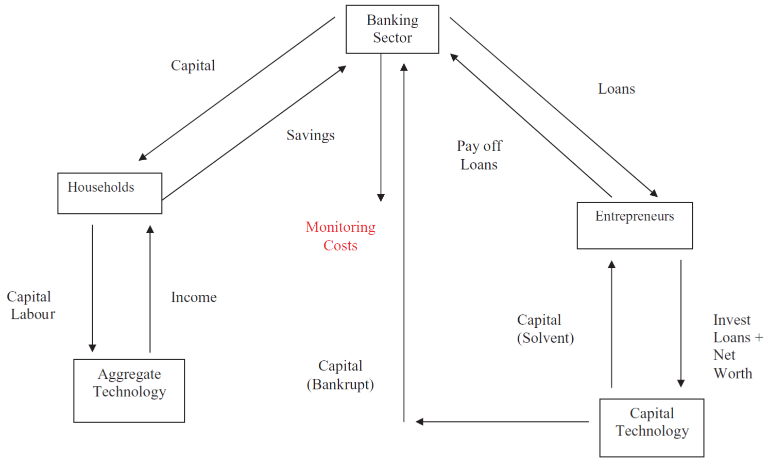
However, this is at odds with empirical evidence. Financial frictions have been modeled in many different ways. The most common approaches are moral hazard problems, adverse selection and asymmetric information, and monitoring costs. An outstanding example is given by Carlstrom and Fuerst (1997) who introduced financial frictions into an otherwise standard neoclassical growth model and showed how the frictions may affect the dynamic properties of the model.

To create the financing friction, Carlstrom and Fuerst (1997) introduce a new type of household, an “Entrepreneur”. This agent is endowed with a technology that converts output into new capital goods. However, he can increase his return even more by borrowing additional resources from a representative, competitive bank so that he can produce more capital than his own resources permit.

The relationship with the lender is modelled assuming asymmetric information between entrepreneurs and banks and a costly state verification as in Townsend (1979) and Gale and Hellwig (1985). Each entrepreneur purchases unfinished capital from the capital producers at the given price and transforms it into finished capital with a technology that is subject to idiosyncratic productivity shocks. The idiosyncratic shocks are assumed to be independently and identically distributed (*i.i.d.*) across entrepreneurs and time. Moreover, the idiosyncratic shock to entrepreneurs is private information for the entrepreneur. To observe this, the lender must pay an auditing cost that is a fixed proportion $\mu \in [0,1]$ of the realized gross return to capital held by the entrepreneur. The optimal loan contract will induce the entrepreneur to not misreport his earnings and will minimize the expected auditing costs incurred by the lender. Under these assumptions, the optimal contract is a standard debt with costly bankruptcy. If the entrepreneur does not default, the lender receives a fixed payment independent of the realization of the idiosyncratic shock; in contrast, if the entrepreneur defaults, the lender audits and seizes whatever it finds.

Here is a very brief description of the model (graphically presented in Figure 1).

Figure 1.1 Sketch of the model



Source. This chart is taken from Dorofeenko, Lee, and Salyer (2008) (pag. 378).

The model is a variant of a standard RBC model in which an additional production sector is added. This sector produces capital using a technology that transforms investment into capital. In a standard RBC framework, this conversion is always one-to-one; in the Carlstrom and Fuerst framework, the production technology is subject to technology shocks. (The aggregate production technology is also subject to technology shocks as is standard.) This capital production sector is owned by entrepreneurs who finance their production via loans from a risk-neutral financial intermediation sector – this lending channel is characterized by a loan contract with a fixed interest rate. (Both capital production and the loans are intra-period.) If a capital-producing firm realizes a low technology shock, it will declare bankruptcy and the financial intermediary will take over production; this activity is subject to monitoring costs. The timing of events is as follows:

1. The exogenous state vector of output technology shocks is realized.

2. Firms hire inputs of labour and capital from households and entrepreneurs and produce output via an aggregate production function

3. Households make their labour, consumption and savings/investment decisions. The household transfers consumption goods to the banking sector

4. With the savings resources from households, the banking sector provides loans to entrepreneurs' via the optimal financial contract. The contract is defined by the size of the loan and a cutoff level of productivity for the entrepreneurs' capital-creation technology shock

5. Entrepreneurs use their net worth and loans from the banking sector as inputs into their capital-creation technology.

6. The idiosyncratic capital-creation technology shock of each entrepreneur is realized. If productivity is large enough, the entrepreneur is solvent and the loan from the bank is repaid; otherwise the entrepreneur declares bankruptcy and production is monitored by the bank at a given cost

7. Entrepreneurs that are solvent make consumption choices; these in part determine their net worth for the next period.

The purpose of these notes is to describe the Carlstrom-Fuerst model in detail. A similar model with financial frictions, where the frictions lie with the buyers of capital—instead of the producers of capital, as it is here—may be found in Bernanke, Gertler, and Gilchrist (1999). Section 2 focuses on the optimal financial contract, while Section 3 explained how to plug the optimal contract in a general equilibrium framework. Few comments (very important!) on the notation: upper case variables denote aggregate quantities while lower case denote per-capita quantities.

Chapter 2

The optimal financial contract

The entrepreneur enters period t with $h_t^e = 1$ unit of labour endowment and k_t^e units of capital. Labour is supplied inelastically while capital is rented to firms; hence income in the period is $w_t^e + r_t k_t^e$. This income along with remaining (i.e., not depreciated) capital determines net worth (denoted as n_t and denominated in units of consumption) at time t:

$$n_t = w_t^e + k_t^e(r_t + q_t(1-\delta)).$$

An Entrepreneur with net worth n_t, who wants to invest i_t has to borrow:

$$b_t = i_t - n_t$$

from the bank. Under the contract, the Entrepreneur agrees to pay back $(1 + r_t^b)(i_t - n_t)$ consumption goods—i.e., loan principal plus the lending interest rate. The Entrepreneur who invests i_t consumption goods into the capital-creation technology draws an idiosyncratic shock, ω, and produces new capital goods in the amount, $i_t\omega$

$$\text{capital goods} = \text{f(consumption goods)}$$

These goods are sold at market price, q_t, so that the value of the consumption good $i_t\omega$ is $q_t i_t \omega$. Note here that ω is *i.i.d.* with probability density function $\phi(\omega)$; cumulative density function $\Phi(\omega)$; positive support, $\omega \geq 0$; and expected value equal to one, $E(\omega) = 1$. The random variable, ω, is realized after the Entrepreneur invests the amount i_t. After ω is realized, only the Entrepreneur knows its value. For an outsider to observe ω they must pay a monitoring cost, which is assumed to be a fraction of the invested amount (as discussed below).

According to the contract, if the Entrepreneur experience a shock ω which is bad enough to make the repayment $(1 + r_t^b)(i_t - n_t)$ infeasible, then he must declare bankruptcy and repay whatever he has, namely, $i_t\omega$.

Definition 1 *There is a cutoff value of the idiosyncratic shock, $\bar{\omega}_t$, such that for all $\omega \leq \bar{\omega}_t$ it is* *infeasible for the Entrepreneur with net worth n_t to repay his loan. It satisfies:*

$$(1 + r_t^b)(i_t - n_t) - q_t \bar{\omega}_t i_t = 0 \quad \Longleftrightarrow \quad \bar{\omega}_t \equiv \tfrac{(1+r_t^b)(i_t - n_t)}{q_t i_t} \tag{2.1}$$

That is:

$\omega_t \geq \bar{\omega}_t$ Entrepreneur repays his debt and enjoys the profits
$\omega_t < \bar{\omega}_t$ Entrepreneurs defaults and looses everything

When an Entrepreneur declares bankruptcy, the bank verifies this by monitoring the Entrepreneur. If the bank did no monitoring, the Entrepreneur would have an incentive to under-report the value of ω, and repay only a small amount to the bank. But, monitoring is expensive. The bank must expend μi_t units of capital goods to monitor an Entrepreneur.

The optimal borrowing contract is given by an amount of borrowing and a lending rate that maximizes the entrepreneur's profits subject to the lender's supply schedule. In the next two subsections we analyze the lender's supply schedule and the maximization problem, respectively. Before getting there, it is useful to recap some definitions

n_t	Net worth of entrepreneurs (in consumption units)
i_t	Investment of entrepreneurs (in consumption units)
$b_t = i_t - n_t$	Loans to entrepreneurs (in consumption units)
i_t / n_t	Leverage (ℓ_t)
$(i_t - n_t)/i_t$	$\equiv (\ell_t - 1)/\ell_t$
ω	idiosyncratic shock with with p.d.f. $\phi(\omega)$, c.d.f. $\Phi(\omega)$; and, $E(\omega) = 1$
$i_t \omega$	Amount of capital goods produced with an investment i_t
r_t^b	Lending rate
$\bar{\omega}_t$	Cutoff value of ω (values below $\bar{\omega}_t$ make entrepreneurs bankrupt)
q_t	Price of capital goods

2.1 Banks zero profit condition – Capital supply

The source of funds for the bank is the household. It is assumed that the banking sector is competitive and that the bank pays the household a zero net rate of return (which implies a gross rate of return equal to 1). That is, if the bank obtains $i_t - n_t$ units of output goods from the household, it must return the same amount in period t. This is just a harmless normalization, we can introduce a non–zero interest rate for households lending activity.

Competitiveness implies that banks are subject to a zero–profit condition, that is:

$$\mathbb{E}\left[\text{Income}\right] = \mathbb{E}\left[\text{Costs}\right]$$

The average (or expected) income of the financial intermediary, integrating across all entrepreneurs is:

$$\begin{aligned}
& \underbrace{\int_{\bar{\omega}_t}^{\infty}(1+r_t^b)(i_t-n_t)\Phi(d\omega)}_{\text{Income (no default)}} + \underbrace{q_t i_t\left(\int_0^{\bar{\omega}_t}\omega\Phi(d\omega) - \int_0^{\bar{\omega}_t}\mu\Phi(d\omega)\right)}_{\text{Income (default)}} \\
= \;& (1+r_t^b)(i_t-n_t)[1-\Phi(\bar{\omega}_t)] + q_t i_t\left(\int_0^{\bar{\omega}_t}\omega\Phi(d\omega) - \mu\Phi(\bar{\omega}_t)\right) \\
= \;& q_t i_t \bar{\omega}_t[1-\Phi(\bar{\omega}_t)] + q_t i_t\left(\int_0^{\bar{\omega}_t}\omega\Phi(d\omega) - \mu\Phi(\bar{\omega}_t)\right) \\
= \;& q_t i_t\left(\int_0^{\bar{\omega}_t}\omega\Phi(d\omega) - \mu\Phi(\bar{\omega}_t) + \bar{\omega}_t[1-\Phi(\bar{\omega}_t)]\right) \\
= \;& q_t i_t g(\bar{\omega}_t)
\end{aligned}$$

While the cost is simply given by the $i_t - n_t$ units of output goods obtained from the household (remember that the risk–free interest rate is set to zero). Therefore:

$$\underbrace{q_t i_t g(\bar{\omega}_t)}_{\text{Expected income}} = \underbrace{1\cdot(i_t-n_t)}_{\text{Expected cost}} \tag{2.2}$$

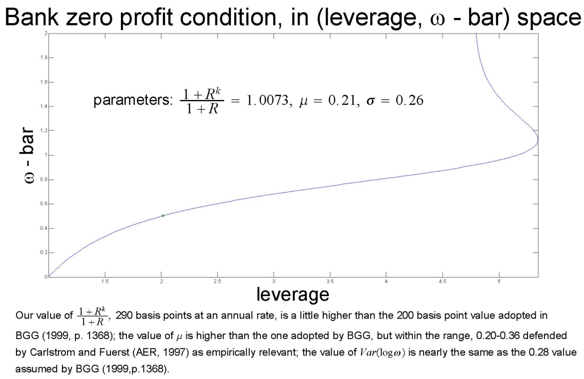
Definition 2 *The gross risk free interest rate (normalized to 1 here) is equated to "banks' average return on entrepreneurial projects".*[1]

Note also that we can rewrite the above expression as:

$$q_t g(\bar{\omega}_t) = \frac{\ell_t - 1}{\ell_t}$$

which allows to draw the capital supply schedule in the space $(\bar{\omega}_t, \ell_t)$. In particular, the zero–profit curve represents a "menu" of contracts that can be offered in equilibrium. Notice that only the upward–sloped portion of the curve is relevant, because entrepreneurs would never select a high value of $\bar{\omega}_t$ if a lower one was available at the same leverage.

[1] This is a source of inefficiency in the model. A benevolent planner would prefer that the market price savers correspond to the marginal return on projects (see Christiano-Ikeda).



Source. This chart is taken from Christiano (2005)

2.2 Entrepreneurs maximization problem – Capital demand

Entrepreneurs maximize expected profits. Therefore, first we have to find an expression for the average (or expected) net profits across all entrepreneurs who invest i_t[2]:

$$
\begin{aligned}
& \underbrace{q_t i_t \int_{\bar{\omega}_t}^{\infty} \omega \Phi(d\omega)}_{\text{Revenues (no default)}} - \underbrace{\int_{\bar{\omega}_t}^{\infty} (1+r_t^b)(i_t - n_t)\Phi(d\omega)}_{\text{Repayment to banks (no default)}} \\
= \; & q_t i_t \int_{\bar{\omega}_t}^{\infty} \omega \Phi(d\omega) - \int_{\bar{\omega}_t}^{\infty} q_t i_t \bar{\omega}_t \Phi(d\omega) \\
= \; & q_t i_t \left(\int_{\bar{\omega}_t}^{\infty} \omega \Phi(d\omega) - \bar{\omega}_t \int_{\bar{\omega}_t}^{\infty} \Phi(d\omega) \right) \\
= \; & q_t i_t \left(\int_{\bar{\omega}_t}^{\infty} \omega \Phi(d\omega) - \bar{\omega}_t [1 - \Phi(\bar{\omega}_t)] \right) \\
= \; & q_t i_t f(\bar{\omega}_t)
\end{aligned}
$$

The Entrepreneur's expected rate of return in the capital producing technology must be no less than zero, because he can always earn a zero return by simply holding onto n_t and not producing capital goods. So, the participation constraint of the Entrepreneur is:

$$q_t i_t f(\bar{\omega}_t) \geq n_t.$$

Competition ensures that, in equilibrium, the debt contract maximizes entrepreneurial profits subject to banks zero–profit condition (2.2). Entrepreneur will maximize:

$$
\begin{aligned}
& \max_{\bar{\omega}_t, i_t} q_t i_t f(\bar{\omega}_t) \qquad (2.3) \\
& s.t. \quad q_t i_t g(\bar{\omega}_t) \geq i_t - n_t.
\end{aligned}
$$

[2]Note here that when the Entrepreneurs defaults his profits are zero: whatever he produces is seized by the bank (which assumes the loss).

The associated Lagrangian is:

$$\Xi = q_t i_t f(\bar{\omega}_t) - \lambda \left(q_t i_t g(\bar{\omega}_t) - i_t + n_t\right).$$

In this problem, q_t and n_t are treated as given, reflecting the assumption that banks are competitive. The first order conditions of this problem are:

$$\begin{aligned}
FOC(\bar{\omega}_t): &\quad q_t i_t f'(\bar{\omega}_t) - \lambda q_t i_t g'(\bar{\omega}_t) = 0 \\
FOC(i_t): &\quad q_t f(\bar{\omega}_t) - \lambda \left(q_t g(\bar{\omega}_t) - 1\right) = 0 \\
FOC(\lambda): &\quad q_t i_t g(\bar{\omega}_t) - i_t + n_t = 0
\end{aligned}$$

By combining the first two equations to substitute out λ:

$$\begin{aligned}
q_t f(\bar{\omega}_t) &= \frac{f'(\bar{\omega}_t)}{g'(\bar{\omega}_t)} \left(q_t g(\bar{\omega}_t) - 1\right) \\
i_t &= \frac{1}{1 - q_t g(\bar{\omega}_t)} n_t
\end{aligned} \tag{2.4}$$

From the we first equation of (2.4) we can pin down the cutoff value of the idiosyncratic shock ($\bar{\omega}_t$) for each given level of price of capital (q_t). From the second equation of (3.1) we can pin down the optimal value for investment i_t, for each given level of net worth (n_t) and price of capital (q_t). Note from (2.4) that the cutoff value of the capital technology shock is a function of the price of capital and of the distribution of ω:

$$\bar{\omega} = f(q, \Phi(\cdot))$$

and not of the level of net worth of the Entrepreneur. On the contrary, the optimal amount of investment is a function of net worth, the price of capital, and of the distribution of ω:

$$i = f(n, q, \Phi(\cdot)).$$

Once the cutoff value of the idiosyncratic shock and the optimal amount of investment are determined, the amount of borrowing is also pinned down. From (2.1) we can back out the level of the interest rate paid by entrepreneurs to banks.

Definition 3 *The lending rate paid by non-bankrupt entrepreneurs is:*

$$(1 + r_t^b) \equiv \frac{q_t i_t \bar{\omega}_t}{(i_t - n_t)} \tag{2.5}$$

A loan to an individual Entrepreneur is risky, in that it may not be repaid fully, and in the event that it is not the bank must incur monitoring costs. So, a natural measure of the risk premium is the excess of $(1 + r_t^b)$ over the sure rate of return, which in this case is unity.

Definition 4 *The risk premium of the lending rate on the risk free rate is:*

$$(1 + r_t^b) - 1 \equiv \frac{q_t i_t \bar{\omega}_t}{(i_t - n_t)} - 1 \tag{2.6}$$

We just showed that under the standard debt contract—i.e., the one that solves the maximization problem in (2.3)—, the level of investment (and, therefore, of loans) that an Entrepreneur can operate is proportional to his net worth. A consequence of this is that in working out the aggregate implications of the model, we do not have to keep track of the distribution of net worth across entrepreneurs. Although that distribution is non-trivial, we can simply work with i_t and n_t, which we interpret as the average, across all entrepreneurs, of investment and net worth, respectively.

To compute the optimal values of $\bar{\omega}_t$, i_t/n_t, and $(1 + r_t^b)$ we have to derive expressions for $f'(\bar{\omega}_t)$ and $g'(\bar{\omega}_t)$; to make assumptions about $\Phi(\omega)$ in order to get numerical expressions for $f(\bar{\omega}_t)$ and $g(\bar{\omega}_t)$.

Chapter 3

Getting $f'(\bar{\omega}_t)$ and $g'(\bar{\omega}_t)$

Using Leibniz's rule we get:

$$\begin{aligned} f'(\bar{\omega}) &= \frac{\partial}{\partial \bar{\omega}} \left(\int_{\bar{\omega}_t}^{\infty} \omega \Phi(d\omega) - \bar{\omega}_t [1 - \Phi(\bar{\omega}_t)] \right) \\ &= -\bar{\omega}\Phi'(\bar{\omega}) - (1 - \Phi(\bar{\omega})) + \bar{\omega}\Phi'(\bar{\omega}) = -(1 - \Phi(\bar{\omega})) \end{aligned}$$

and:

$$\begin{aligned} g'(\bar{\omega}) &= \frac{\partial}{\partial \bar{\omega}} \left(\int_{0}^{\bar{\omega}_t} \omega \Phi(d\omega) - \mu\Phi(\bar{\omega}_t) + \bar{\omega}_t [1 - \Phi(\bar{\omega}_t)] \right) \\ &= \bar{\omega}\Phi'(\bar{\omega}) - \mu\Phi'(\bar{\omega}) + (1 - \Phi(\bar{\omega})) - \bar{\omega}\Phi'(\bar{\omega}) = -\mu\Phi'(\bar{\omega}) + (1 - \Phi(\bar{\omega})) \end{aligned}$$

so that the first order conditions in (2.4) reduce to:

$$\begin{aligned} q_t f(\bar{\omega}_t) &= \frac{1}{\mu \frac{\Phi'(\bar{\omega})}{1-\Phi(\bar{\omega})} - 1} (q_t g(\bar{\omega}_t) - 1) \\ i_t &= \frac{1}{1 - q_t g(\bar{\omega}_t)} n_t \end{aligned} \tag{3.1}$$

From the second equation of (3.1), we can see how much investment, i_t, an Entrepreneur with net worth, n_t, can do. Notice that this expression is equivalent to equation 3.8 of BGG (page 1353):

> "Equation (3.8) describes the critical link between capital expenditures by the firm and financial conditions, as measured by the wedge between the expected the return to capital and the safe rate, s_t, and by entrepreneurial net worth, N^j_{t+1}"

3.1 Getting $f(\bar{\omega}_t)$ and $g(\bar{\omega}_t)$

Assume ω is log–normally distributed (then $x = \log(\omega)$ is normally distributed).

BOX. Log–normal Distribution. In probability theory, If X is a random variable with a normal distribution, then $Y = exp(X)$ has a log-normal distribution; likewise, if Y is log-normally distributed, then $X = log(Y)$ has a normal distribution.
The log-normal distribution is the distribution of a random variable that takes only positive real values. The probability density function of a log-normal distribution is

$$f_X(x;\mu;\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma^2}\right)^2},$$

and the cumulative distribution function is

$$\Phi = \left(\frac{\ln x - \mu}{\sigma}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution.
If X is a lognormally distributed variable, its expected value (E, which can be assumed to represent the arithmetic mean) and variance (Var) are:

$$\begin{aligned} E[X] &= e^{\mu + \frac{1}{2}\sigma^2} \\ Var[X] &= (e^{\sigma^2} - 1)e^{2\mu + \sigma^2} \end{aligned}$$

Equivalently, parameters μ and σ can be obtained if the expected value and variance are known:

$$\begin{aligned} \mu &= \ln\left(E[X]) - \frac{1}{2}\sigma^2\right) \\ \sigma^2 &= \ln\left(1 + \frac{Var[X]}{(E[X])^2}\right) \end{aligned}$$

Therefore, using the fact that:

$x = \log(\omega) \sim \mathcal{N}(-\frac{1}{2}\sigma_x^2, \sigma_x^2)$	so that $E[\omega] = 1$
$\frac{d\omega}{dx} = \frac{de^x}{dx} = e^x$	change the integration variable

and making the following change of variable $\omega = e^x$, we get:

$$\begin{aligned}
\int_0^{\bar{\omega}} \omega d\Phi(\omega) &= \int_0^{\bar{\omega}} \omega\phi(\omega)d\omega = \ ... \\
&= \int_0^{\bar{\omega}} e^x\phi(e^x)dxe^x = \text{ made the change of variable} \\
&= (0 < \omega < \bar{\omega} \Longleftrightarrow 0 < e^x < \bar{\omega} \Longleftrightarrow -\infty < e^x < \log(\bar{\omega})) = \text{change of support} \\
&= \int_{-\infty}^{\log(\bar{\omega})} e^x \frac{1}{e^x\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\ln e^x + \frac{1}{2}\sigma_x^2\right)^2}{\sigma^2}} e^x dx = \text{ used the definition of density function for log–normal} \\
&= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\log(\bar{\omega})} e^x e^{-\frac{1}{2}\frac{\left(x + \frac{1}{2}\sigma_x^2\right)^2}{\sigma^2}} dx = \text{ rearranged} \\
&= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\log(\bar{\omega})} e^{-\frac{1}{2}\frac{\left(x - \frac{1}{2}\sigma_x^2\right)^2}{\sigma^2}} dx = \text{ combined the powers of } e \text{ and rearrange the square products}
\end{aligned}$$

Now define the variable:

$v = \frac{x+\frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x = \frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x}$	
$\frac{dx}{dv} = \frac{d(v\sigma_x+\frac{1}{2}\sigma_x^2)}{dv} = \sigma_x$	change the integration variable

and get:

$$\begin{aligned}
\int_0^{\bar{\omega}} \omega d\Phi(\omega) &= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\log(\bar{\omega})} e^{-\frac{1}{2}v^2}\sigma_x dv = \\
&= \left(-\infty < x < \log(\bar{\omega}) \Longleftrightarrow -\infty < v\sigma_x + \frac{1}{2}\sigma_x^2 < \log(\bar{\omega}) \Longleftrightarrow -\infty < v < \frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x} e^{-\frac{1}{2}v^2} dv =
\end{aligned}$$

Notice that the last is the density function of the standard normal distribution:

$$\int_0^{\bar{\omega}} \omega d\Phi(\omega) = \Pr\left[v < \frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x\right].$$

Therefore, we can compute:

$$g(\bar{\omega}_t) = \int_0^{\bar{\omega}_t} \omega\Phi(d\omega) - \mu\Phi(\bar{\omega}_t) + \bar{\omega}_t[1 - \Phi(\bar{\omega}_t)]$$

Finally, notice that, for each unit of investment, $f(\bar{\omega}_t)$ is the fraction of expected net capital output going to entrepreneurs, while $g(\bar{\omega}_t)$ is the fraction of expected net capital output received by the lender:

$$
\begin{aligned}
& g(\bar{\omega}_t) + f(\bar{\omega}_t) \\
= \; & \int_0^{\bar{\omega}_t} \omega \Phi(d\omega) - \mu\Phi(\bar{\omega}_t) + \bar{\omega}_t[1 - \Phi(\bar{\omega}_t)] + \int_{\bar{\omega}_t}^{\infty} \omega\Phi(d\omega) - \bar{\omega}_t[1 - \Phi(\bar{\omega}_t)] \\
= \; & 1 - \mu\Phi(\bar{\omega}_t)
\end{aligned}
\tag{3.2}
$$

which implies that so that, on average, $\mu\Phi(\bar{\omega}_t)$ of produced capital is destroyed in monitoring. From this last expression we can back out the value of $f(\bar{\omega}_t)$.

3.2 Getting Φ'

With the same notation as above we can show that:

$$
\begin{aligned}
\Phi(\bar{\omega}) &= \int_0^{\bar{\omega}} d\Phi(\omega) = \int_0^{\bar{\omega}} \phi(\omega) d\omega = \\
&= \int_{-\infty}^{\bar{x}} \phi(e^x) dx e^x = \int_{-\infty}^{\bar{x}} \frac{1}{e^x \sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\ln e^x + \frac{1}{2}\sigma_x^2\right)^2}{\sigma^2}} e^x dx = \\
&= \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\bar{x}} e^{-\frac{1}{2}\frac{\left(x + \frac{1}{2}\sigma_x^2\right)^2}{\sigma^2}} dx
\end{aligned}
$$

which is the normal cumulative distribution for $x \sim \mathcal{N}(-\frac{1}{2}\sigma_x^2, \sigma_x^2)$. Moreover, we can compute the derivative of the above with Leibniz's rule, which yields:

$$
\Phi'(\bar{\omega}) = \frac{1}{\bar{\omega}\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x^2}}
$$

which is the standard normal probability density function pre–multiplied by a constant.

Chapter 4

General Equilibrium Model

Carlstrom and Fuerst (1997) showed how to integrate the above debt arrangement into an otherwise standard version of the neoclassical growth model. We the above notation in mind, let's go through the timing of the model one more time. Notice that is assumed that the economy is composed of firms, a mass η of entrepreneurs and a mass $1-\eta$ of identical households.

4.1 Model and timing

1. The exogenous technology shock, denoted by (θ_t), is realized
2. Households and entrepreneurs supply their labor, $H = (1-\eta)l_t$ and $H^e = \eta \cdot 1$, respectively. They earn competitive wage rates, w_t and w_t^e, respectively.
3. Also, households supply k_t^c and entrepreneurs supply their average stock of capital, k_t^e. So, total beginning-of-period t capital, K_t, supplied to the capital-rental market is:

$$K_t = (1-\eta)k_t^c + \eta k_t^e$$

   Households and entrepreneurs earn the competitive rental rate, r_t, on their capital supply.
4. Final output, Y_t, is produced by goods-producing firms using a technology that is homogeneous in capital, household labor and entrepreneurial labor:

$$Y_t = F(K_t, \theta_t, H_t, H_t^e).$$

5. Households allocate their income to consumption (c_t) and savings of capital goods $(k_{t+1}^c - (1-\delta)k_t^c)$, that they transfer to banks. In doing so, they supply to the bank

$$q_t[k_{t+1}^c - (1-\delta)k_t^c] = q_t\,(i_t - n_t)$$

   They require only a zero net return on these deposits (normalization! could be different form zero)

6. The average income of entrepreneurs is $w_t^e + r_t k_t^e$ in units of consumption. The average value of their un–depreciated capital is $q_t(1-\delta)k_t^e$, in units of consumption. Therefore, at this point, the average value (in consumption units) of the entrepreneurs' net worth (i.e. resources) is:
$$n_t = w_t^e + [r_t + q_t(1-\delta)]\, k_t^e \tag{4.1}$$
Equation (4.1) is the law of motion of Entrepreneurs' net worth. Entrepreneurs invest all their net worth plus what they borrow from the bank into the production of capital goods.

7. The idiosyncratic technology shock of each entrepreneur is realized. If $\omega^j \geq \bar{\omega}$ the entrepreneur is solvent and the loan from the bank is repaid; otherwise the entrepreneur declares bankruptcy and production of capital goods is monitored by the bank at a cost of μi_t.

8. At the end of the period, after the debt contract with the bank is paid off, the entrepreneurs who do not go bankrupt in the process of producing capital have income that can be used to buy consumption goods and new capital goods:
$$c_t^e + q_t k_{t+1}^e \leq \begin{cases} (1+r_t^b)(i_t - n_t) - i_t\omega & \omega \geq \bar{\omega} \\ 0 & \omega < \bar{\omega} \end{cases}$$
An Entrepreneur who is bankrupt in period t must set $c_t^e = 0$ and $k_{t+1}^e = 0$. In period $t+1$, the net worth of bankrupt entrepreneurs is their wage bill w_{t+1}^e. Now, one can see why it is assumed that entrepreneurs earn wage income. An Entrepreneur with no assets cannot borrow anything from the bank, and zero assets would become an absorbing state. Entrepreneurs who are not bankrupt in period t, instead, can purchase positive amounts of c_e^t and k_{t+1}^e (except in the non-generic case, $\omega = \bar{\omega}$).

4.2 Households

The household problem is:

$$\max_{\{c_t, k_{t+1}^c\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{4.2}$$
$$\begin{aligned} s.t. \quad & c_t + q_t\,(i_t - n_t) \leq w_t l_t + r_t k_t^c \\ & (i_t - n_t) = k_{t+1}^c - (1-\delta)k_t^c \end{aligned}$$

and to an initial level of capital, k_0^c. Here, c_t and k_t^c denote household consumption and the household stock of capital, respectively. In addition, l_t denotes household employment and $q_t\,(i_t - n_t)$ households' savings. From the problem defined in (4.2) we can construct the Lagrangian as:

$$\Xi = \sum_{t=0}^{\infty} (\beta)^t \left(u(c_t, l_t) - \lambda_t \left[c_t + q_t \left[k^c_{t+1} - (1-\delta) k^c_t \right] - w_t l_t - r_t k^c_t \right] \right)$$

Therefore we can compute the first order conditions:

$$\begin{array}{ll} FOC(c_t): & u_{c,t} - \lambda_t = 0, \\ FOC(l_t): & u_{l,t} - \lambda_t w_t = 0, \\ FOC(k^c_{t+1}): & -\lambda_t q_t + \beta \lambda_{t+1} \left(q_{t+1}(1-\delta) + r_{t+1} \right) = 0, \\ FOC(\lambda): & c_t + q_t \left[k^c_{t+1} - (1-\delta) k^c_t \right] - w_t l_t - r_t k^c_t = 0. \end{array} \tag{4.3}$$

The optimal intertemporal condition (Euler equation) is:

$$q_t u_{c,t} = \beta u_{c,t+1} \left[q_{t+1}(1-\delta) + r_{t+1} \right],$$

and the optimal intratemporal condition is:

$$\frac{u_{l,t}}{u_{c,t}} = w_t.$$

4.3 Firms

The economy's output is produced by firms using Cobb–Douglas technology. Firms are competitive and they maximize profits:

$$\max_{k_t, l_t, l^e_t} Y_t - (r_t K_t) - (w_t H_t) - (w^e_t H^e_t) \tag{4.4}$$

$$s.t. \quad Y_t = F(K_t, \theta_t, H_t, H^e_t)$$

The optimality conditions imply:

$$\begin{aligned} F_{K,t} &= r_t \\ F_{H,t} &= w_t \\ F_{H^e,t} &= w^e_t \end{aligned} \tag{4.5}$$

4.4 Entrepreneurs

A risk-neutral representative entrepreneur's course of action is as follows. To finance his project at period t, he borrows resources from the Capital Mutual Fund according to an optimal financial contract. The entire borrowed resources, along with his total net worth at period t, are then invested into his capital creation project. If the representative entrepreneur is solvent after observing his own technology shock, he then makes his consumption decision; otherwise, he declares bankruptcy and production is monitored (at a cost) by the Capital Mutual Fund.

Entrepreneurs are assumed to have discounted utility:

$$\max_{\{c_t^e, k_{t+1}^e\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} (\gamma\beta)^t c_t^e \tag{4.6}$$
$$s.t. \quad c_t^e + q_t k_{t+1}^e \leq q_t i_t f(\bar{\omega}_t)$$

The budget constraint is the relevant one for the consumption decision: Entrepreneurs have to decide c_t^e and k_{t+1}^e after the realization of the capital technology shock (ω). Their resources at that time are $q_t i_t f(\bar{\omega}_t)$. Note also that entrepreneurs discount utility at a higher rate, $\gamma\beta$, than households ($0 < \gamma < 1$). This new parameter, γ , will be chosen so that it offsets the steady-state internal rate of return to entrepreneurs' investment. The entrepreneurs are endowed with one unit of labor time, which they supply inelastically to the market. From the problem defined in (4.6) we can construct the Lagrangian as:

$$\Xi = \sum_{t=0}^{\infty} (\gamma\beta)^t \left(c_t^e - \lambda_t \left[c_t^e + q_t k_{t+1}^e - q_t i_t f(\bar{\omega}_t)\right]\right)$$

where we know that i_t is given by the solution of the optimal contract in (2.4):

$$\Xi = \sum_{t=0}^{\infty} (\gamma\beta)^t \left(c_t^e - \lambda_t \left[c_t^e + q_t k_{t+1}^e - q_t \frac{1}{1 - q_t g(\bar{\omega}_t)} n_t f(\bar{\omega}_t)\right]\right)$$

and n_t is given by (4.1):

$$\Xi = \sum_{t=0}^{\infty} (\gamma\beta)^t \left(c_t^e - \lambda_t \left[c_t^e + q_t k_{t+1}^e - q_t \frac{1}{1 - q_t g(\bar{\omega}_t)} \left(w_t^e + \left[r_t + q_t(1-\delta)\right] k_t^e\right) f(\bar{\omega}_t)\right]\right).$$

Therefore we can compute the first order conditions:

$$\begin{aligned}
&FOC(c_t^e): && 1 - \lambda_t = 0 \\
&FOC(k_{t+1}^e): && -q_t\lambda_t + \gamma\beta\lambda_{t+1}\left(q_{t+1}\tfrac{1}{1-q_{t+1}g(\bar{\omega}_{t+1})}\left(r_{t+1} + q_{t+1}(1-\delta)\right) f(\bar{\omega}_{t+1})\right) = 0 \\
&FOC(\lambda): && c_t^e + q_t k_{t+1}^e - q_t i_t f(\bar{\omega}_t) = 0
\end{aligned} \tag{4.7}$$

Combining $FOC(c_t^e)$ and $FOC(k_{t+1}^e)$ we get:

$$q_t = \gamma\beta \left(\left[r_{t+1} + q_{t+1}(1-\delta)\right] \cdot \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})} \right)$$

which is the Euler equation for entrepreneurs (as in the last equation of Carlstrom and Fuerst, pag. 898). Notice that the term on the right of the multiplication sign is the "expected return to internal funds", as defined in Carlstrom and Fuerst paper.

Definition 5 *The expected return to internal funds is $q_{t+1} f(\bar{\omega}_{t+1}) i_t / n_t$ i.e., the net worth of size n_t is leveraged into a project of size i_t , entrepreneurs keep the share $f(\bar{\omega}_{t+1})$ of the capital produced and capital is priced at q_t consumption goods. Since these are intra-period loans, the opportunity cost is 1.*

The expression to the left of "·" coincides with the rate of return enjoyed by households. As explained above, the expression to the right of "·" must be no less than unity. (The Entrepreneur can always obtain unity, simply by not producing any capital.) In this expression, we see why it is assumed that entrepreneurs discount the future more heavily than households do. Entrepreneurs earn a higher intertemporal rate of return on saving than do households. As a result, entrepreneurs with the same discount rate as households would save at a higher rate, eventually accumulating enough capital (and, hence, net worth) so that they have no need to borrow from banks. The assumption, $\gamma < 1$, helps ensure that the financial frictions remain operative indefinitely in this economy.

4.5 Banks

The Capital Mutual Funds (CMFs) act as risk-neutral financial intermediaries who earn no profit and produce neither consumption nor capital goods. There is a clear role for the CMF in this economy since, through pooling, all aggregate uncertainty of capital production can be eliminated. The CMF receives capital from three sources: entrepreneurs sell undepreciated capital in advance of the loan; after the loan, the CMF receives the newly created capital through loan repayment and through monitoring of insolvent firms; and, finally, those entrepreneurs that are still solvent sell some of their capital to the CMF to finance current period consumption. This capital is then sold at the price of q_t units of consumption to households for their investment plans.

Chapter 5

Equilibrium

The 15 variables to be determined : $c_t, c_t^e, i_t, K_t, k_t^e, l_t, q_t, \bar{\omega}_t, \theta_t, r_t, w_t, w_t^e, Y_t, H_t, H_t^e$. Therefore, we need 15 equations. By combining (4.5) with (4.3) we get Households' intertemporal condition (Euler equation), which is also households demand for capital

$$q_t u_{c,t} = \beta u_{c,t+1} \left[q_{t+1}(1-\delta) + r_{t+1} \right], \tag{5.1}$$

and Households' intertemporal condition

$$\frac{u_{l,t}}{u_{c,t}} = w_t. \tag{5.2}$$

The budget constraint of entrepreneurs (per capita) is

$$k_{t+1}^e = i_t f(\bar{\omega}_t) - \frac{c_t^e}{q_t}. \tag{5.3}$$

Entrepreneurs net worth evolution (per capita) is

$$n_t = w_t^e + \left[r_t + q_t(1-\delta) \right] k_t^e. \tag{5.4}$$

The aggregate accumulation of capital is

$$K_{t+1} = (1-\delta)K_t + I_t[1 - \mu\Phi(\bar{\omega}_t)], \tag{5.5}$$

where notice that aggregate investment is $I_t = \eta i_t$. The aggregate resource constraint is

$$\underbrace{(1-\eta)c_t}_{C_t^c} + \underbrace{\eta c_t^e}_{C_t^e} + \underbrace{\eta i_t}_{I_t} = Y_t. \tag{5.6}$$

The Entrepreneurs' intertemporal condition (Euler equation) is

$$\frac{1}{\gamma\beta} = \left(\frac{F_{1,t} + q_{t+1}(1-\delta)}{q_t} \cdot \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})} \right). \tag{5.7}$$

By combining (3.1) with (3.2), we get the contract efficiency condition (the equation for the determination of the optimal cutoff value of the idiosyncratic shock)

$$q_t = \frac{1}{1 - \mu\Phi(\bar{\omega}) + \frac{\mu\Phi'(\bar{\omega})f(\bar{\omega}_t)}{1-\Phi(\bar{\omega})}} \tag{5.8}$$

Finally, Entrepreneurs' (per capita) capital demand is

$$i_t = \frac{1}{1 - q_t g(\bar{\omega}_t)} n_t. \tag{5.9}$$

The aggregate output is given by

$$Y_t = F(K_t, \theta_t, H_t, H_t^e), \tag{5.10}$$

the rental rate of capital

$$r_t = F_{K,t}, \tag{5.11}$$

the consumers' wage

$$w_t = F_{l,t}, \tag{5.12}$$

and entrepreneurs wage is

$$w_t^e = F_{H^e,t}. \tag{5.13}$$

Finally, as explained above, Households aggregate labor is given by

$$H_t = (1-\eta) l_t. \tag{5.14}$$

Entrepreneurs aggregate labor is given by

$$H_t^e = \eta. \tag{5.15}$$

5.1 Parametrization

Assume that utility is logarithmic in c_t and linear in l_t and has the form

$$u(c_t, l_t) = \ln(c_t) - v(1 - l_t)$$

where the constant $v = 2.52$ is chosen so that steady state aggregate labor is $H = l(1-\eta) = 0.3$. This implies

$$\begin{aligned} u_c &= \frac{1}{c_t} \\ u_l &= v \end{aligned}$$

The production function has the following form

$$Y = K_t^{\alpha}\left(\theta_t H_t^{\zeta}\right)\left(H_t^{e}\right)^{(1-\alpha-\zeta)}$$

implying that

$$\begin{aligned}
r_t &= \alpha K_t^{\alpha-1}\left(\theta_t H_t^{\zeta}\right)\left(H_t^{e}\right)^{(1-\alpha-\zeta)} \\
w_t &= \zeta K_t^{\alpha}\left(\theta_t H_t^{\zeta-1}\right)\left(H_t^{e}\right)^{(1-\alpha-\zeta)} \\
w_t^{e} &= (1-\alpha-\zeta) K_t^{\alpha}\left(\theta_t H_t^{\zeta}\right)\left(H_t^{e}\right)^{(-\alpha-\zeta)}
\end{aligned}$$

They assign a share of 0.36 to capital (i.e., α = 0.36) and a share of 0.6399 and 0.0001 to household employment and entrepreneurial employment, respectively (i.e., $\zeta = 0.0001$). The small share of employment by entrepreneurs implies their wage rate is very small, though they still earn enough of a wage so that bankrupt entrepreneurs can finance at least some investment. Because the share of income going to entrepreneurial labor is so small, when $\mu = 0$ the economy essentially collapses to the real business cycle model—note from (5.8) that $q = 1$ in this case). Carlstrom and Fuerst set $\delta = 0.02$ and $\beta = 0.99$. They also set $\mu = 0.25$.

To obtain the parameters of the log-normal distribution, Carlstrom and Fuerst suppose, first, that $E[\omega] = 1$. There now remain two parameters to set: γ and σ. The latter is the standard deviation of the normal random variable, $\log(\omega)$. These two parameters were pinned down by specifying values for the bankruptcy rate in steady state, $\Phi(\bar{\omega})$, and the risk premium on loans to entrepreneurs, $(1 + r_t^b) - 1$. Carlstrom and Fuerst specify the annualized risk premium to be 187 basis points (i.e., 1.87 percentage points) and a quarterly bankruptcy rate of 0.974 percent.

$$\frac{1}{\beta} = \frac{q(1-\delta)+r}{q}$$

In addition, note that from (5.1) and (5.7), the entrepreneur's intertemporal problem is

$$\begin{aligned}
\frac{1}{\gamma\beta} &= \underbrace{\frac{q(1-\delta)+r}{q}}_{1/\beta \text{ from HH Steady state}} \cdot \frac{q f(\bar{\omega})}{1 - q g(\bar{\omega})} \\
\frac{1}{\gamma} &= \frac{q f(\bar{\omega})}{1 - q g(\bar{\omega})}
\end{aligned}$$

Given the target risk premium and the latter equation, Carlstrom and Fuerst report $\sigma = 0.207$ and $\gamma = 0.947$.

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